Stefan's law

It states that —

The total amount of heat radiated by a perfectly black body per second per unit area is directly proportional to fourth power of its absolute temperature. i.e,

$$E \propto T^4$$

$$E = \sigma T^4$$

The value of σ in M.K.S unit is $5.67 \times 10^{-8}\ W m^{-2} K^{-4}$.

The law can be extended to represent the net loss of heat by the body after exchange with the surroundings. and enunciated as follows

"If a black body at absolute temperature T is surrounded by another black body at absolute temperature T_0, the amount of heat lost by the former per second per square centimetre is given by

$$E = \sigma (T^4 - T_0^4)$$

The law can be written as

$$E = \sigma e T_0^4$$

where e is the emissivity of the body
The value of e lies between zero and one, depending upon the nature of the surface. For a perfectly black body $e = 1$.

Proof :-

According to Quantum theory of radiation a photon consists of an energy $h\nu$, where ν is the frequency of radiation and h the Planck's constant.

If m is the mass of the photon, then according to the theory of relativity

$$h\nu = mc^2$$

$$mc = \frac{h\nu}{c} \qquad \text{——— (1)}$$

Where c is the velocity of light. The ~~equation (1)~~

If n is the number of photons contained in a cylinder of length c and unit area of cross-section, then total momentum imparted by these photons incident normally on a unit area per second is given by

$$\text{Momentum imparted} = \frac{nh\nu}{c}$$

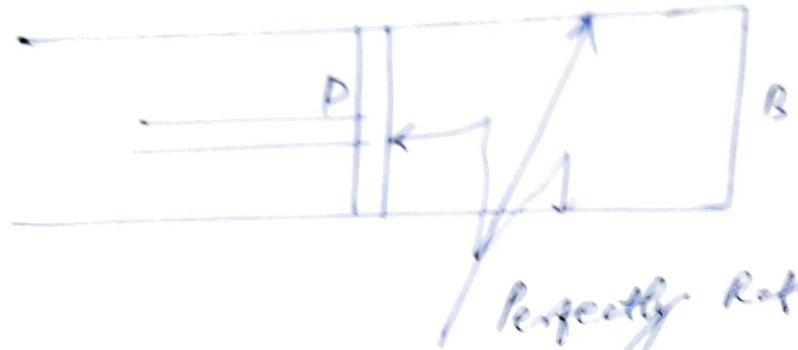
The pressure is the force per unit area or the momentum imparted per unit area.

$$\therefore \quad \text{Pressure } p = \frac{nh\nu}{c} = \text{Energy density } E$$

If the incident radiation is diffused i.e. it is incident in all direction, then

$$\text{pressure } p = \frac{1}{3}E \qquad \text{——— (2)}$$

Now, consider a cylinder of unit area of cross-section having perfectly reflecting walls and the piston. The base B of the cylinder is perfectly black. Suppose the cylinder is filled with radiation at a uniform temperature T and let a small quantity of radiation dQ be incident on the box from outside due to which the temperature rise by dT. This results in the increase of internal energy



If E is the energy density of radiation and V is the volume, then

Internal energy $= VE$

$\therefore$ Change in internal energy $= d(VE)$

$= VdE + EdV$

If the piston is allowed to move so that the volume increases by a small value dV, against a constant pressure p of the radiation inside the cylinder.

then from ②, we have

External work done $= pdV = \frac{1}{3} E \cdot dV$

The energy dQ is partly used in increasing the internal energy and partly in doing external work.

$$\therefore dQ = (VdE + EdV) + \frac{1}{3} EdV$$

$$= VdE + \frac{4}{3} EdV$$

If dS is the ~~energy~~ change in entropy of radiation then

$$dQ = TdS$$

$$\therefore\ TdS = VdE + \frac{4}{3}EdV$$

or $$dS = \frac{V}{T}\cdot dE + \frac{4}{3}\frac{E}{T}dV \quad \text{———— (3)}$$

Now, dS is a perfect differential $\quad S = f(E, V)$

$$\therefore\ dS = \frac{\partial S}{\partial E}\cdot dE + \frac{\partial S}{\partial V}dV \quad \text{———— (4)}$$

Comparing (3) & (4), we have

$$\frac{\partial S}{\partial E} = \frac{V}{T} \quad \text{and} \quad \frac{\partial S}{\partial V} = \frac{4}{3}\frac{E}{T}$$

$$\therefore\ \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial E}\right) = \frac{\partial}{\partial E}\left(\frac{\partial S}{\partial V}\right)$$

or $$\frac{\partial}{\partial V}\left(\frac{V}{T}\right) = \frac{\partial}{\partial E}\left(\frac{4}{3}\frac{E}{T}\right)$$

Now temperature T is independent of V and is a function of E alone

$$\therefore\ \frac{1}{T} = \frac{4}{3}\left(\frac{1}{T} - \frac{1}{T^2}E\cdot\frac{\partial T}{\partial E}\right)$$

$$1 = \frac{4}{3} - \frac{4}{3}\frac{E}{T}\cdot\frac{\partial T}{\partial E}$$

$$\frac{1}{3} = \frac{4}{3}\frac{E}{T}\frac{\partial T}{\partial E}$$

or $$\frac{\partial E}{E} = \frac{4\partial T}{T}$$

Integrating both sides.

$$\log E = 4\log T + C$$

or $$\boxed{E = \sigma T^4}$$